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Appendix A. The general conditions for diffusion instabiltiy to arise are reviewed. It is shown that predator-prey models given by Eqs. 4 and 5 in the paper with prey-dependent functional response cannot give rise to Turing structures.

Conditions for diffusion instability

In order to examine the conditions for diffusion instability to arise (Okubo 1980, Murray 1989), the following predator-prey equations are used:

$$\frac{dN}{dt} = f(N)N - g(N,P)P + D_N \frac{\partial^2 N}{\partial x^2} \qquad \text{(A.1)}$$

$$\frac{dP}{dt} = eg(N,P)P - \mu P + D_P \frac{\partial^2 P}{\partial x^2}, \qquad \text{(A.2)}$$

which can be summarized in a more general form:

$$\frac{\partial N}{\partial t} = F(N,P) + D_N \frac{\partial^2 N}{\partial x^2} \qquad \text{(A.3)}$$

$$\frac{\partial P}{\partial t} = G(N,P) + D_P \frac{\partial^2 P}{\partial x^2}. \qquad \text{(A.4)}$$

This system will display diffusion-driven instability if there is a spatially uniform state where preys and predators coexist in a stable equilibrium that becomes unstable to certain spatially inhomogeneous small perturbations. This property can be outlined analytically by means of three conditions:

1. A feasible coexistence equilibrium point must exist. Thus, the system

$$F(N^*, P^*) = 0 \tag{A.5}$$

$$G(N^*, P^*) = 0 \tag{A.6}$$

must have a feasible solution, $N^* > 0$ and $P^* > 0$.

2. The coexistence point must be stable when subjected to spatially homogeneous small perturbations from the spatially uniform state. To assess stability, the so-called community matrix (Levins 1968) must be evaluated at the equilibrium point (N^*, P^*):

$$\mathbf{J}^* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$$a_{11} = \left.\frac{\partial F}{\partial N}\right|_{(N^*,P^*)} \quad a_{12} = \left.\frac{\partial F}{\partial P}\right|_{(N^*,P^*)}$$

$$a_{21} = \left.\frac{\partial G}{\partial N}\right|_{(N^*,P^*)} \quad a_{22} = \left.\frac{\partial G}{\partial P}\right|_{(N^*,P^*)}. \tag{A.7}$$

If the trace of $\mathbf{J}^*$ is negative while its determinant is positive the equilibrium point will be stable in front of spatially homogeneous small perturbations:

$$a_{11} + a_{22} < 0 \tag{A.8}$$

$$a_{11}a_{22} - a_{21}a_{12} > 0 \tag{A.9}$$

3. The coexistence stable point (N^*, P^*) must be unstable when subjected to inhomogeneous small perturbations. The condition for that to occur depends also on the elements of the community matrix, and on the relative diffusion, $d = D_P / D_N$. It can be expressed as (Murray 1989):

$$d\, a_{11} + a_{22} > 2\sqrt{d}\sqrt{\det(\mathbf{J}^*)} \tag{A.10}$$

The fulfillment of the three conditions ensures the generation of spatial pattern through diffusion-driven instability.

Prey-dependent models: absence of Turing structures

In prey-dependent models, the predator functional response, $g(N)$, does not depend on predator abundance. Therefore, the general equations (A.1)-(A.2) become:

$$\frac{\partial N}{\partial t} = f(N)N - g(N)P + D_N \frac{\partial^2 N}{\partial x^2} \tag{A.11}$$

$$\frac{\partial P}{\partial t} = eg(N)P - \mu P + D_P \frac{\partial^2 P}{\partial x^2}. \tag{A.12}$$

It has already been shown in previous studies (Segel and Jackson 1972) that the system Eqs. (A.11) and (A.12) cannot present diffusion instability. The reason can be easily understood if the entries of the community matrix are evaluated at the equilibrium point, e.

g., $N^* = g^{-1}(\mu/e)$, $P^* = e/\mu\, N^* f(N^*)$:

$$a_{11} = f(N^*) + N^* \left.\frac{df}{dN}\right|_{N^*} - \left.\frac{dg}{dN}\right|_{N^*} P^* \tag{A.13}$$

$$a_{12} = -g(N^*) = -\frac{\mu}{e} \tag{A.14}$$

$$a_{21} = e \left.\frac{dg}{dN}\right|_{N^*} P^* \tag{A.15}$$

$$a_{22} = eg(N^*) - \mu = 0 \tag{A.16}$$

From Eq (A.6), it can be seen that a_{22} must be null. Thus, stability condition (A.8) will become $a_{11} < 0$, while condition needed for diffusion-driven instability to occur (A.10) will become $a_{11} > \frac{2\sqrt{d}}{d}\sqrt{\det(\mathrm{J}^*)} > 0$. Therefore, the two conditions cannot be fulfilled simultaneously.

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